Four Special Number Systems

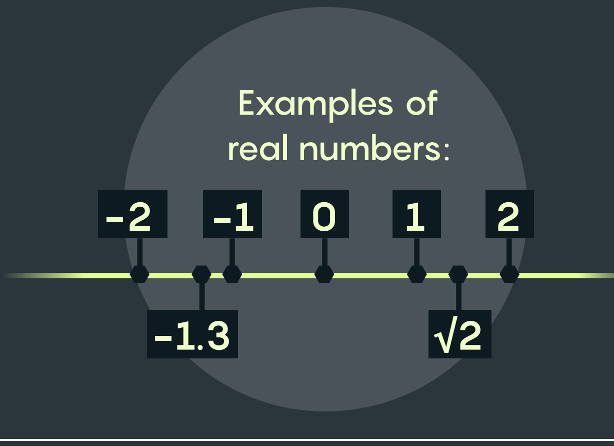
When you add, subtract, multiply or divide the “real numbers” used in everyday life, you always get another real number. Three generalizations of the real numbers also behave in this way. Many physicists believe that all four of these “division algebras” underlie the laws of physics.

ℝ Real numbers

All the numbers on (1-D) number line.

One defining characteristic of reals is that **their square is never negative.**

e.g. $2^2 = 4$ and also $(-2)^2 = 4$



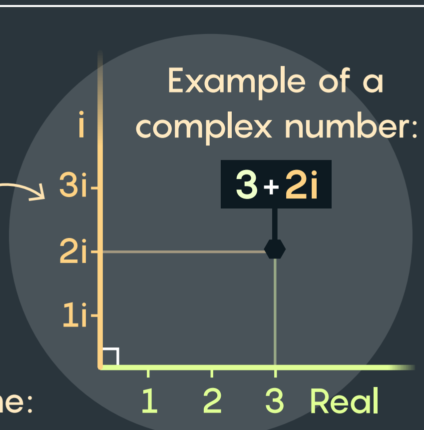
ℂ Complex numbers

Reals used in conjunction with an unconventional “imaginary” unit called i.

One defining characteristic of i is that **its square is negative.**

ie. $i^2 = -1$

Complex numbers behave like coordinates on a 2-D plane:

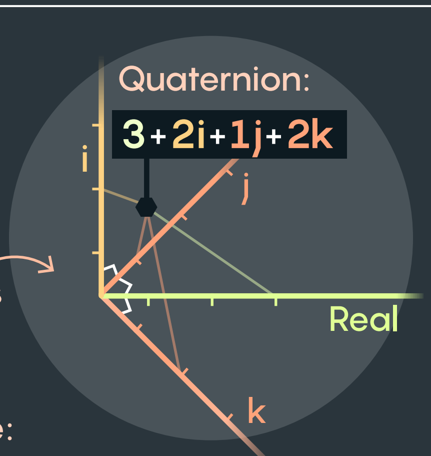


ℍ Quaternions

Reals used in conjunction with three unconventional units called i, j and k.

Multiplication of quaternions is **noncommutative:** Swapping the order of elements changes the answer.

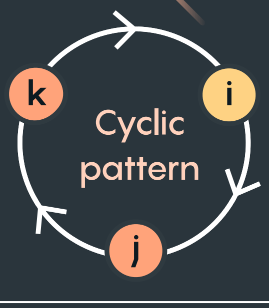
Quaternions behave like coordinates in 4-D space:



Multiplication follows a cyclic pattern, where multiplying neighboring elements results in the third:

Moving with arrows gives a **positive** answer: $i \times j = k$

Moving against arrows gives a **negative** answer: $j \times i = -k$

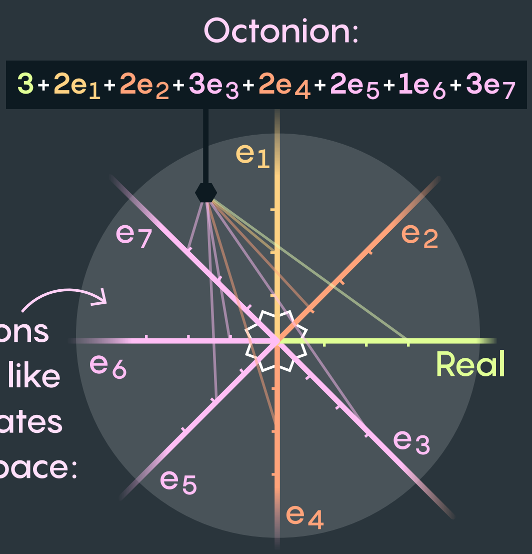


𝕆 Octonions

Reals used in conjunction with seven unconventional units: e_1, e_2, e_3, e_4, e_5, e_6 and e_7 (e_1, e_2 and e_4 are comparable to the quaternions’ i, j and k).

Multiplication of octonions is **nonassociative** — it matters how they are grouped.

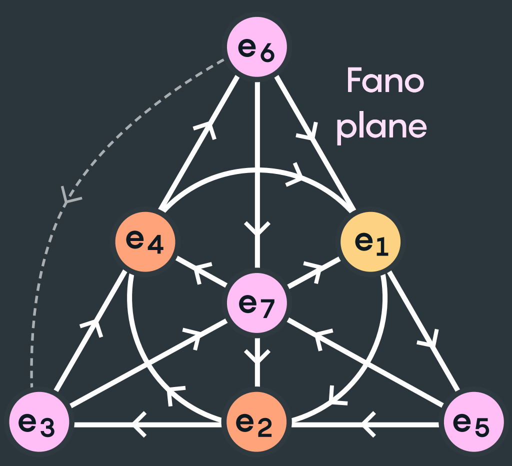
Octonions behave like coordinates in 8-D space:



Their multiplication rules are encoded in the “Fano plane.” Multiplying two neighboring elements on a line results in the third element on that same line. Imagine additional lines that close the loop for each group of three elements (e.g., the dashed line).

Moving with arrows gives a **positive** answer:
e.g. $e_5 \times e_2 = e_3$ and $e_6 \times e_3 = e_4$

Moving against arrows gives a **negative** answer:
e.g. $e_1 \times e_7 = -e_3$ and $e_6 \times e_5 = -e_1$



To see their nonassociative property, multiply three elements e_5, e_2, e_4

$$\left.\begin{array}{ll} \text{Grouping them like this ...} & (e_5 \times e_2) \times e_4 = (e_3) \times e_4 = e_6 \\ \text{But grouping them like this ...} & e_5 \times (e_2 \times e_4) = e_5 \times (e_1) = -e_6 \end{array}\right\} \text{Different answers}$$

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